Quantum Coding Theory	(UC Berkeley CS294, Spring 2024)

Lecture 3: More Noise

January 31, 2024

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1 Noise and Quantum Channels

1.1 General model of noise

The general model of noise we use in quantum computing is given by a quantum channel, which we typically call $\mathcal{N}(\rho)$. In the Kraus representation:

$$\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger \ s.t \ \sum_i E_i E_i^\dagger = \mathbb{I}.$$

You can think of each E_i as specifying an error on the state and the channel $\mathcal{N}(\rho)$ as giving a mixture of these errors in which we output:

$$\frac{E_i \rho E_i^\dagger}{P_{E_i|\rho}} \ w.p. \ P_{E_i|\rho} = tr(E_i \rho E_i^\dagger).$$

In this class, we often look at cases when ρ is a pure state $|\psi\rangle\langle\psi|$. $\mathcal{N}(|\psi\rangle\langle\psi|)$ in this case outputs:

$$\frac{1}{P_{E_i|\psi}} E_i |\psi\rangle \ \ w.p. \ P_{E_i|\psi)} = \langle\psi| E_i^\dagger E_i |\psi\rangle .$$

1.2 More one-qubit of channels

Last time we looked at some channels you might see including the dephasing and depolarizing channel; however unlike the depolarizing channel, noise is typically biased. To deal with this we introduce the general Pauli channel.

General Pauli Channels:

$$\mathcal{N}(\rho) = P_X X\rho X + P_Y Y\rho Y + P_Z Z\rho Z + (1 - P_X - P_Y - P_Z)\rho$$

If you have a qc you might need to spend time to benchmark each P_i, but in doing so you can get a better error-correcting code.

Measurement Channel: given $|\psi\rangle = a|0\rangle + b|1\rangle$

$$\mathcal{N}(|\psi\rangle\langle\psi|) = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$$

Here there are two Kraus operators: $E_0 = |0\rangle\langle 0|$ and $E_1 = |1\rangle\langle 1|$. This is an example of a case where Kraus operators are non-unitary and the probability of E_0 or E_1 is dependent on the state. In this case though, you can remove this dependence as the measurement channel is the same as the dephasing channel with $p = 1/2$: $\mathcal{N}(\rho) = \frac{1}{2}\rho + \frac{1}{2}Z\rho Z$.

Erasure Channel:

$$\mathcal{N}(\rho) = (1-p)\rho + p\,|e\rangle\langle e|$$

Here $|e\rangle$ denotes an erasure symbol or a state that is not a part of the basis. Types of errors like this are nice to work with, as you know exactly where the error is, due to the error message. An example where you might see this error is if your qubits are ions where $|0\rangle$ is the ground state and $|1\rangle$ is the 1st excited state. There is a chance in computation that a qubit gets excited to the second excited state which we can denote by $|e\rangle$, as it is not in our $|0\rangle, |1\rangle$ basis. Another example is if your qubit is a photon and it leaves/disappears (leakage error).

Amplitude Damping Channel:

$$\mathcal{N}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$
$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix},\ E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

If it is in the $|0\rangle$ state then nothing happens, but if it is in the $|1\rangle$ state then there is a chance that it decays to the $|0\rangle$ state. This is more easily seen in the Stinespring representation of a quantum channel:

$$U|0\rangle|0\rangle_E = |0\rangle|0\rangle_E$$
$$U|1\rangle|0\rangle_E = \sqrt{1-\gamma}\,|1\rangle|0\rangle_E + \sqrt{\gamma}\,|0\rangle|1\rangle_E$$

Here, again, the error you have depends on the input state. An example where you might see this error is if your qubits are ions where $|0\rangle$ is the ground state and $|1\rangle$ is the 1st excited state. Then, there is a chance that the excited state might decay into the ground state.

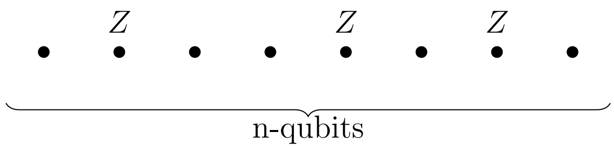
1.3 n-qubit channels

In this class, we care about n-qubit channels. The most basic of which is an independent channel. In QEC you can get channels that aren't independent, but looking at independent ones first helps before going further.

Def: $\mathcal{N}$ is an independent channel if $\mathcal{N} = \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \cdots \otimes \mathcal{E}_n$

ex:

$$\mathcal{N} = \mathcal{E}^{\otimes n} \qquad \mathcal{E}(\rho) = (1-p)\rho + pZ\rho Z \quad (\textit{dephasing channel})$$



expect $\approx p \cdot n$ errors

Because each error is independent you expect to not see more than $p \cdot n$ errors due to the Chernoff bound.

Chernoff Bound:
Let $X_1, X_2, \cdots, X_n \in \{0,1\}$ be independent r.v.'s such that $Pr[X_i = 1] = p$. If $X_i = 1$, you can think of it as there is an error on qubit i.
Let $S = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[S] = p \cdot n$, by linearity of expectation. Then, the Chernoff bound states:

$$Pr[S \geq \mu + \delta n] \leq \frac{1}{e^{2n\delta^2}} \quad \text{where } \delta \text{ is a parameter we choose}$$

Ex: if $p = \frac{1}{2}$ then the probability that we don't get $\delta = 0.01n$ more than $\frac{1}{2}$ is:

$$Pr[S \geq \frac{n}{2} + 0.01n] \leq \frac{1}{e^{2n \cdot 0.01^2}} = \frac{1}{e^{\theta(n)}}$$

The probability that the number of errors is significantly bigger than the expected value is exponentially unlikely. So, we will just assume going forward that the number of errors is bounded by the expectation. With the channel in the example, sometimes there is an error, sometimes not, but they can also do other things. Sometimes, an error always happens, but it is small.
With the rotation channel:

$$R_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

If θ is small and $R_\theta \approx \mathbb{I}$ except with a small error, you can simulate it as only having an error on a small number of qubits.

Def :

- $A_1 \otimes \cdots \otimes A_n$ has weight t if all but t of $A_i's = \mathbb{I}$ (most will be $\mathbb{I}$, but if $\leq t$ of them are errors then it has weight t).
- $B = \sum_i B_i$ is a t-qubit error if $weight(B_i) \leq t$ for all i.
- $\mathcal{N}(\cdot)$ is a t-qubit error channel if each Kraus operator E_i is a t-qubit error.

2 General errors

2.1 Shore 9-qubit code with general errors

We've already shown that the Shore 9-qubit code works for any arbitrary single Pauli error, but now we will see how it works on all other one-qubit errors. Let's see how it can correct the rotation channel.

$$R_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta & 0 \\ 0 & \cos\theta + i\sin\theta \end{pmatrix} = \cos\theta \cdot \mathbb{I} - i\sin\theta \cdot Z$$

$$\begin{aligned} (R_\theta)_k \left|\psi\right\rangle_L &= (\cos\theta \cdot \mathbb{I} - i\sin\theta \cdot Z)_k \left|\psi\right\rangle_L \\ &= \cos\theta \left|\psi\right\rangle_L - i\sin\theta \cdot Z_k \left|\psi\right\rangle_L \end{aligned}$$

$$\xrightarrow{\substack{\text{error detection} \\ \text{algorithm}}} \quad cos\theta \left|\psi\right\rangle_L \left|\text{"}no\ error\text{"}\right\rangle - isin\theta \cdot Z_k \left|\psi\right\rangle \left|\text{"}Z\ error\ on\ qubit\ k\text{"}\right\rangle$$

There are two things we can do going forward:

Strategy 1 (measure syndrome):
w.p. $cos^2\theta$ measure "no error" and collapse the state to $\left|\psi\right\rangle_L$ or w.p. $sin^2\theta$ measure "Z error on qubit k" and collapse the state to $Z_k \left|\psi\right\rangle_L$
In this case, we had a general error but just ran the algorithm like we had a Z error. In the end, measuring the syndrome collapsed the state into $\mathbb{I}\left|\psi\right\rangle_L$ or $Z\left|\psi\right\rangle_L$. If it happens to collapse to the logical state where the Z error was applied, we can fix it because we know the result of the syndrome measurement.

Strategy 2 (error correct coherently):

$$cos\theta \left|\psi\right\rangle_L \left|\text{"}no\ error\text{"}\right\rangle - isin\theta \cdot Z_k \left|\psi\right\rangle \left|\text{"}Z_k\ error\text{"}\right\rangle$$

$$\begin{aligned} \xrightarrow{\text{correct}} \quad & cos\theta \left|\psi\right\rangle_L \left|\text{"}no\ error\text{"}\right\rangle - isin\theta \left|\psi\right\rangle \left|\text{"}Z_k\ error\text{"}\right\rangle \\ &= \left|\psi\right\rangle_L \otimes (cos\theta \left|\text{"}no\ error\text{"}\right\rangle - isin\theta \left|\text{"}Z_k\ error\text{"}\right\rangle) \end{aligned}$$

Here the syndrome part is independent of the logical state, so we discard it and are just left with $\left|\psi\right\rangle_L$. For more general errors think of the syndrome as a vector (in this case $(cos\theta \left|\text{"}no\ error\text{"}\right\rangle - isin\theta \left|\text{"}Z_k\ error\text{"}\right\rangle)$ is the vector).

2.2 General Errors

We can correct general errors by expressing it as a linear combination of Pauli errors.
General one qubit error:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+d}{2} + \frac{a-d}{2} & \frac{b+c}{2} + \frac{b-c}{2} \\ \frac{b+c}{2} - \frac{b-c}{2} & \frac{a+d}{2} - \frac{a-d}{2} \end{pmatrix}$$

$$= (\frac{a+d}{2}) \cdot \mathbb{I} + (\frac{a-d}{2}) \cdot Z + (\frac{b+c}{2}) \cdot X + (\frac{b-c}{2}) \cdot iY$$

Fact: Pauli matrices span all 2×2 matrices $\therefore$ if we can correct all single-qubit Pauli errors, we can correct all one-qubit errors. This is often called the discretization\digitization of errors.
This means that you only need to correct a finite number of errors. If you can do this then you can correct anything those errors span.

Def: given a code space C and Hilbert space $\mathcal{H}$

1. A quantum error correcting code (QECC) is a subspace $C \subseteq \mathcal{H}_{physical}$
2. An encoding map is a unitary $U : \mathcal{H}_{logical} \to C$

   e.g. $\mathcal{H}_{logical} = (\mathbb{C}^2)^{\otimes k}, \quad \mathcal{H}_{physical} = (\mathbb{C}^2)^{\otimes n}, \quad C = span\{|\bar{x}\rangle\}_{x \in \{0,1\}^k}$

3. A set of errors $\mathcal{E}$ is corrected by C if there exists a quantum algorithm $Rec(\cdot)$ s.t. $\forall E \in \mathcal{E}$, $|\psi\rangle_L \in C$, $Rec(\frac{1}{\sqrt{P_{E|\psi}}} E \cdot |\psi\rangle_L = |\psi\rangle_L)$

Let $\mathcal{N}$ be a quantum channel with Kraus operators $E_i \in \mathcal{E}$.

$$\mathcal{N}(|\psi\rangle_L \langle\psi|_L) = \text{output} \frac{1}{\sqrt{P_{E_i|\psi}}} E_i |\psi\rangle_L \text{ w.p. } P_{E_i|\psi}$$

$$\xrightarrow{\text{Rec}(\cdot)} \qquad |\psi\rangle_L$$

The algorithm $Rec(\cdot)$ can be represented as a quantum circuit as in Figure 1.

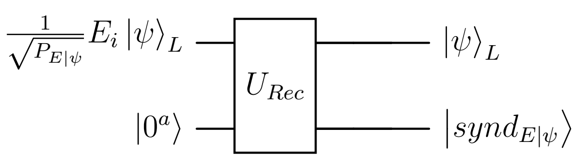


Figure 1: Here is the Stinespring representation of the channel $Rec(\cdot)$

The top row is our input and output values, while the bottom row is the ancilla qubits. This ancilla register is not completely junk as it records the error; however, this error may depend on ψ.

Fact: Let $\mathcal{E}$ be the set of errors on C that $Rec(\cdot)$ corrects. Then $\mathcal{E}$ is a linear subspace (of matrices).

Proof: Let $E_1, E_2 \in \mathcal{E}$

$$\begin{aligned} & \frac{1}{\sqrt{P_{E_1+E_2|\psi}}}(E_1+E_2)\left|\psi\right\rangle_L = \alpha\frac{1}{\sqrt{P_{E_1|\psi}}}E_1\left|\psi\right\rangle_L + \beta\frac{1}{\sqrt{P_{E_2|\psi}}}E_2\left|\psi\right\rangle_L \\ \xrightarrow{\mathrm{U}_{Rec}} \quad & \left|\psi\right\rangle_L \otimes \left(\alpha\left|synd_{E_1|\psi}\right\rangle + \beta\left|synd_{E_2|\psi}\right\rangle\right) \\ & = \left|\psi\right\rangle_L \left|synd_{E_1+E_2|\psi}\right\rangle \square \end{aligned}$$